Limits at Infinity

Motivating Example: A small pond has an oxygen concentration of 12 units. Suppose at time $t=0$ organic waste is introduced into the pond. The oxygen concentration t weeks later is

$$f(t) = \frac{12t^2 - 15t + 12}{t^2+1}$$

Q/: In the long run what will happen to the oxygen concentration?

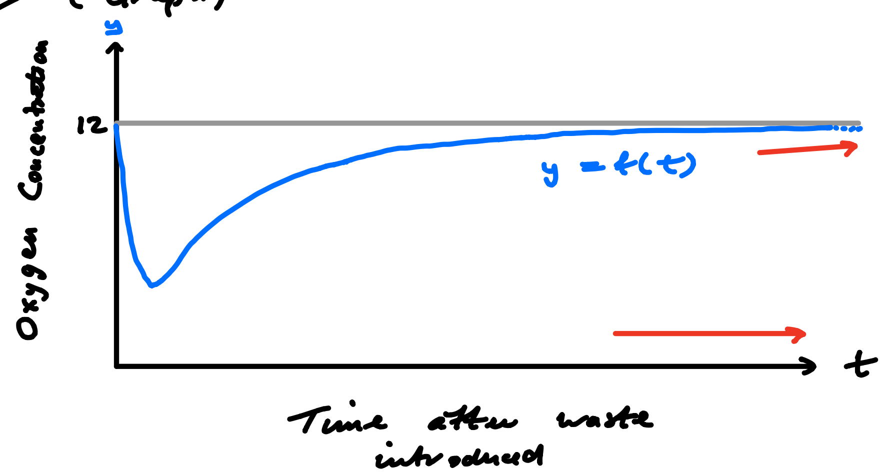
1/ (Table)

t grows positively without bound

t	10	1000	100000
$f(t)$	10.515	11.985	11.99985

$f(t)$ approaches 12

2/ (Graph)



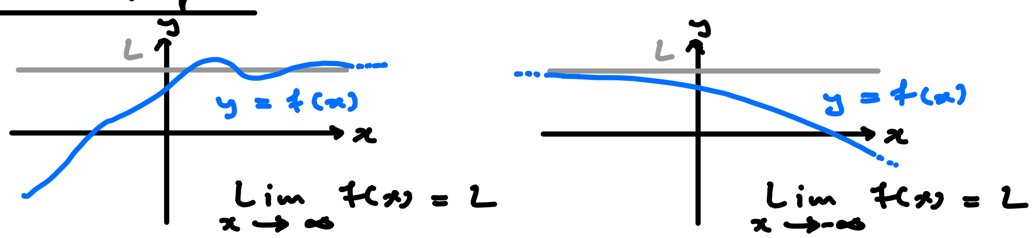
Conclusion As t grows positively without bound, $f(t)$ approaches 12.

Definition (Limits at Infinity)

$\lim\limits_{x \to \infty} f(x) = L \iff f(x)$ approaches L as x grows positively without bound

$\lim\limits_{x \to -\infty} f(x) = L \iff f(x)$ approaches L as x grows negatively without bound

Basic Graphs:



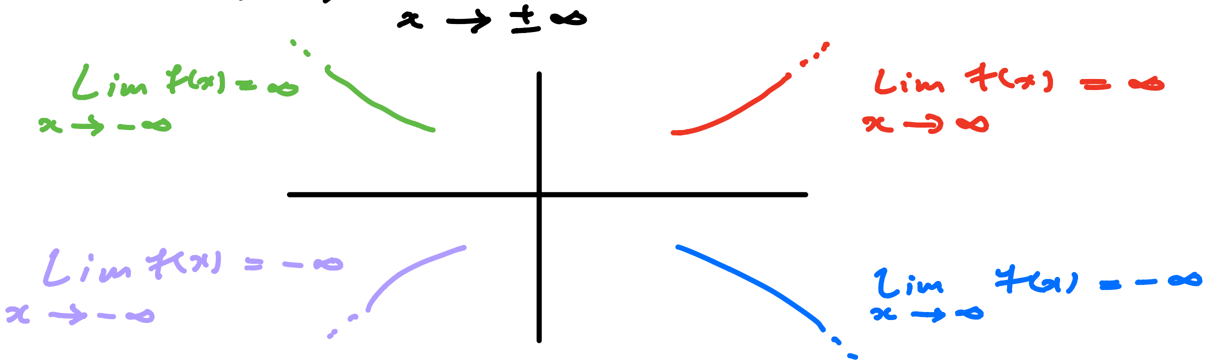
Examples $\lim\limits_{x \to \pm\infty} \frac{1}{x^n} = 0$ $(n>0)$, $\lim\limits_{x \to \infty} b^x = 0$ $(b<1)$.

$\lim\limits_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e = 2.718\ldots$

Remarks 1/ $\lim\limits_{x \to \infty} f(x) \neq f(\infty)$ ← Meaningless ∞ is not a number.

2/ If no such L exists we say "the limit does not exist". (DNE)

For example, $\lim\limits_{x \to \pm\infty} f(x) = \pm\infty$



Examples: $\lim\limits_{x \to \infty} b^x = \infty$ $(b>1)$, $\lim\limits_{x \to \infty} \ln(x) = \infty$,

$$\lim_{x \to \infty} x^r = \infty \quad (r > 0)$$

3/ All the rules for calculating limits we have developed are the same for limits at infinity.

<u>Important Example</u> : $f(x) = \dfrac{p(x)}{q(x)}$ ← polynomials

$$\Rightarrow \lim_{x \to \pm\infty} \frac{p(x)}{q(x)} = \begin{cases} 0 & \text{if } \deg(q) > \deg(p) \\ \text{DNE} & \text{if } \deg(q) < \deg(p) \\ \text{Ratio of leading coefficients} & \text{if } \deg(q) = \deg(p) \end{cases}$$

<u>Examples</u>

1/ $\displaystyle \lim_{x \to \infty} \frac{x^2+3x+1}{x^3+2x+1} = ?$

$\deg(x^3+2x+1) = 3 > \deg(x^2+3x+1)$

$$\Rightarrow \lim_{x \to \infty} \frac{x^2+3x+1}{x^3+2x+1} = 0$$

2/ $\displaystyle \lim_{x \to -\infty} \frac{6x^4-1}{7x^4+2x+1} = ?$

$\deg(7x^4+2x+1) = 4 = \deg(6x^4-1)$

$$\Rightarrow \lim_{x \to -\infty} \frac{6x^4-1}{7x^4+2x+1} = \frac{6}{7}$$

Harder Example

$$\lim_{x\to\infty} \left(\ln(x^2+1) - 2\ln(2x+4)\right) = ?$$

$$\ln(x^2+1) - 2\ln(2x+4) = \ln(x^2+1) - \ln\left((2x+4)^2\right)$$

$$= \ln\left(\frac{x^2+1}{(2x+4)^2}\right) = \ln\left(\frac{x^2+1}{4x^2+16x+16}\right)$$

$$\deg(x^2+1) = \deg(4x^2+16x+16) = 2$$

$$\Rightarrow \lim_{x\to\infty} \frac{x^2+1}{4x^2+16x+16} = \frac{1}{4}$$

$$\Rightarrow \lim_{x\to\infty} \left(\ln(x^2+1) - 2\ln(2x+4)\right) = \ln\left(\frac{1}{4}\right)$$

Very Hard Example :

$$\lim_{x\to 2} e^{\frac{-x}{x^3-4x^2+4x}} = ?$$

if $x \neq 0$

$$\frac{-x}{x^3-4x^2+4x} = \frac{-1}{(x-2)^2}$$

$$\lim_{x\to 2} -1 = -1 < 0$$

$$\lim_{x\to 2} (x-2)^2 = 0^+$$

positive because we are squaring

$$\Rightarrow \lim_{x \to 2} \frac{-1}{(x-2)^2} = -\infty$$

$$\lim_{u \to -\infty} e^{u} = 0 \Rightarrow \lim_{x \to 2} e^{\frac{-1}{(x-2)^2}} = 0$$

Hard part. As x approaches 2 $\frac{-1}{(x-2)^2}$ grows negatively without bound. As we feed in a bigger an bigger negative number $e^{\frac{-1}{(x-2)^2}}$ gets smaller and smaller.